The Solutions to IMO 2007 Problems

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1. Problem I.

(1) Suppose contrary that there exists an n-tuple (x_i) such that$|x_i - a_i| < \frac{d}{2}$ for all $i = 1, 2, \dots, n$. Therefore,

$$x_i - \frac{d}{2} < a_i < x_i - \frac{d}{2}\,,$$

for each $i = 1, 2, \dots, n$.
Thus, for any j, $1 \le j \le i$,

$$a_j < x_j + \frac{d}{2} \le x_i + \frac{d}{2}\,.$$

Similarly, if $i \le j \le n$,

$$a_j > x_j - \frac{d}{2} \ge x_i - \frac{d}{2}\,.$$

This implies

$$\max_{1 \le j \le i} a_j < x_i + \frac{d}{2}$$

and

$$\min_{i \le j \le n} a_j > x_i - \frac{d}{2}\,.$$

Therefore,

$$d_i = \max_{1 \le j \le i} a_j - \min_{i \le j \le n} a_j < \left(x_i + \frac{d}{2}\right) - \left(x_i - \frac{d}{2}\right) = d\,.$$

Hence, $d_i < d$ for all i. Therefore, $\max\limits_{1 \le i \le n} d_i < d$, which contradicts the definition of d.

(2) Choose $x_i = \max\limits_{1 \le j \le i} a_j - \frac{d}{2}$ for all $i = 1, 2, \dots, n$. It is obvious that (x_i) is nondecreasing. Moreover,

$$x_i - a_i \ge x_i - \max_{1 \le j \le i} a_j \ge -\frac{d}{2}\,.$$

Since

$$x_i - a_i = \left(\max_{1 \le j \le i} a_j - \frac{d}{2}\right) - a_i \le \left(\max_{1 \le j \le i} a_j - \min_{i \le j \le n} a_j\right) - \frac{d}{2}\,.$$

Hence,

$$x_i - a_i \leq d_i - \frac{d}{2} \leq d - \frac{d}{2} = \frac{d}{2}\,.$$

Therefore,

$$|x_i - a_i| \leq \frac{d}{2}\,.$$

for all $i = 1, 2, \ldots, n$. From the first part, the equality must hold for this choice of (x_i).

2. Problem II.

We shall prove the following statement instead: the circumcenter E of the triangle CFG lies on the circumcircle of the triangle BCD if and only if ℓ bisects $\angle DAB$.

(1) Suppose that ℓ bisects $\angle DAB$. Note that $BG = AB$ since ABG is isosceles. Moreover, $AB = CD$ since $ABCD$ is a parallellogram. Hence, $BG = CD$. Also,

$$\angle EGC = \frac{\pi}{2} - \frac{\angle CEG}{2} = \frac{\pi}{2} - \angle CFG = \frac{\pi}{2} - \angle BAG = \frac{\pi}{2} - \frac{\angle BAD}{2}\,.$$

Similarly, $\angle ECD = \dfrac{\pi}{2} - \dfrac{\angle BAD}{2}$. Consequently, $\angle EGB = \angle EGC = \angle ECD$. Because E is the circumcenter of CFG, $EG = EC$.
Therefore, $BG = CD$, $\angle EGB = \angle ECD$, and $EG = EC$. That is, the triangles BGE and DCE are congruent. Therefore, DCE is a rotation image of BCE about E. This implies the angles between corresponding sides are equal, namely, $\angle BCD = \angle BED$. This proves the converse.

(2) Suppose that E lies on the circumcenter of BCD. Let M be the intersection of the diagonals AC and BD. If we can show that EM is perpendicular to BD, we prove the assertion. ($EM \perp BD$ implies $\angle EDB = \angle EBD$ since M is the midpoint of BD. If $\angle EDB = \angle EBD$, then $\angle ECD = \angle ECG$ and thus, $\angle GAD = \angle GAB$.)
Let U and V be the midpoint of CF and CG, respectively. Suppose also that the perpendicular from E meets BD at M'. Hence, $EU \perp CF$ and $EV \perp CG$. Since E is on the circumference of the trangle BCD, Simson's theorem says that the endpoints of the altitudes from E are collinear. Hence, U, V, and M' are on the same straight line. However, since U and V are the midpoints of CF and CG, the homothety mapping U and V, consecutively, to F and G must also map M' to A. This implies that M' is the midpoint of AC, and thus, $M' = M$. Therefore, $EM \perp BD$, as desired.

3. Problem III.

Let $G(V, E)$ be a simple graph whose vertices represent the competitors and there is an edge between a pair of contestants iff they are acquainted. Define $\gamma(H)$ to be the size of the largest clique of a graph H and $V(H)$ the set of vertices of H. The hypothesis is that $\gamma(G)$ is even. Therefore, we must show that it is possible to divide G into two disjoint subgraphs G_1 and G_2 such that $V(G_1) \cup V(G_2) = V$ and $\gamma(G_1) = \gamma(G_2)$. We need to prove the following two lemmas first.

Lemma 1: For any partition (G_1, G_2) of G, if we move one vertex of G_1 to G_2, $\gamma(G_1)$ decreases by 0 or 1, while $\gamma(G_2)$ increases by 0 or 1.
Proof:
This is clear. Then, the proof is omitted.

□

Lemma 2: For any partition (G_1, G_2) of G, $\gamma(G_1) + \gamma(G_2) \geq \gamma(G)$.
Proof:
Let $a = \gamma(G_1)$, $b = \gamma(G_2)$, and $c = \gamma(G)$. Consider the largest clique K of G, suppose that k vertices of K lies in G_1. Hence, $k \leq a$. Moreover, the rest $c - k$ vertices of K must be in G_2. Therefore, $c = k + (c - k) \leq a + b$, as desired.

□

We follow the succeeding algorithm. First, let G_1 be the largest clique of G and put all the other vertices into G_2. If $\gamma(G_1) = \gamma(G_2)$, we are done. Otherwise, move one vertex from G_1 continually, but never make $\gamma(G_1) < \gamma(G_2)$. This process must terminate since G_1 has only finitely many vertices. From the first lemma, the difference $\gamma(G_1) - \gamma(G_2)$ can reduce by only 0, 1, or 2. Therefore, we can finally obtain a pair (G_1, G_2) such that $\gamma(G_1) - \gamma(G_2) = 0$ or 1. If $\gamma(G_1) - \gamma(G_2) = 0$, the problem is proved.

Suppose that we arrive at $\gamma(G_1) - \gamma(G_2) = 1$. Let Γ_1, Γ_2, ..., Γ_d be all maximal cliques of G_2. If each in G_1 is adjacent with all vertices in Γ_i for some $i = 1, 2, \ldots, d$, then $\gamma(G) \geq |V(G_1)| + |V(\Gamma_i)|$ (recall that G_1 is always a complete graph). From Lemma 2, $\gamma(G) = |V(G_1)| + |V(\Gamma_i)|$, since $\gamma(G_1) = |V(G_1)|$ and $\gamma(G_2) = |V(\Gamma_i)|$. However, $|V(G_1)| + |V(\Gamma_i)|$ is odd, contradicting with that $\gamma(G)$ is even.

Hence, there exists a vertex v in G_1 such that for all i, $1 \leq i \leq d$, v is not connected to a vertex of K_i. Therefore, moving v into G_2 does not increase $\gamma(G_2)$, while decreasing $\gamma(G_1)$ by 1. This finally makes $\gamma(G_1) = \gamma(G_2)$.

4. Problem IV.

First note that $\dfrac{[RPK]}{[RQL]} = \dfrac{RP \cdot PK}{RQ \cdot QL}$ because $\angle RPK = \angle RQL$. Moreover, $\dfrac{PK}{QL} = \dfrac{CP}{CQ}$ since CPK and CQL are similar. Therefore, $\dfrac{PK}{QL} = \dfrac{PB}{QA}$ (since PK and QL are the perpendicular bisectors of BC and AC, respectively). Thus,

$$\frac{[RPK]}{[RQL]} = \frac{RP \cdot PB}{RQ \cdot QA} = \frac{[RPB]}{[RQA]}\,,$$

since $\angle BPR = \angle ACB = \angle AQR$. However, $BR = AR$, $\angle BPR = \angle AQR$, and

$$\angle BRP = \angle CAB = \pi - \angle ACB - \angle ABC = \pi - \angle AQR - \angle ARQ = \angle RQA$$

imply that the triangles RPB and RQA are equivalent. Hence,

$$\frac{[RPK]}{[RQL]} = \frac{[RPB]}{[RQA]} = 1\,,$$

and we are done.

5. Problem V.

Note that $4ab-1 \mid \left(4a^2-1\right)^2$ implies $4ab-1 \mid \left(\left(4a^2-1\right)-(4ab-1)\right)^2$. Consequently, $4ab-1 \mid (4a)^2(a-b)^2$. Since $\gcd(4ab-1, 4a) = 1$, we arrive at

$$4ab-1 \mid (a-b)^2 .$$

It is easy to see that the conditions $4ab-1 \mid \left(4a^2-1\right)^2$ and $4ab-1 \mid (a-b)^2$ are actually equivalent.

Let S be the set of all pairs of positive integers (a, b) satisfying the conditions such that $a \neq b$. If $S = \emptyset$, we are done. Suppose contrary that $|S| > 0$. Hence, there is a pair of integer a_0 and b_0, $(a_0, b_0) \in S$, whose sum is minimum. Without loss of generality, we assume $a_0 > b_0$.

Let $n = \dfrac{(a_0-b_0)^2}{4a_0b_0-1}$. Hence, the equation

$$\frac{(x-b_0)^2}{4xb_0-1} = n ,$$

or equivalently, $x^2 - (2b_0(2n+1))\,x + \left(b_0^2+n\right) = 0$ has one solution $x_1 = a_0$. The other solution $x_2 = 2b_0(2n+1) - a_0 = \dfrac{b_0^2+n}{a_0}$ is hence a positive integer (since $2b_0(2n+1) - a_0 \in \mathbb{Z}$ and $\dfrac{b_0^2+n}{a_0} > 0$). Notice that $x_2 \neq b_0$; otherwise $n = 0$, which would imply $a_0 = b_0$.

Therefore, both (a_0, b_0) and (x_2, b_0) are elements of S. Since $a_0 + b_0$ is minimum, $x_2 + b_0 \geq a_0 + b_0$. Therefore, $x_2 \geq a_0$, or

$$a_0^2 - b_0^2 \geq a_0x_2 - b_0^2 = \left(b_0^2+n\right) - b_0^2 = n .$$

Consequently,

$$\frac{(a_0-b_0)^2}{4a_0b_0-1} \leq a_0^2 - b_0^2 ,$$

or equivalently,

$$a_0 - b_0 \geq (a_0+b_0)\,(4a_0b_0-1) > a_0 + b_0 ,$$

which is a contradiction. Therefore, the only solution to $4ab-1 \mid \left(4a^2-1\right)^2$ is $a = b$.

6. Problem VI.

Lemma 1: Let F be an arbitrary field and f a polynomial in $F[x_1, x_2, \ldots, x_n]$ such that the degree of f as a polynomial in x_i is at most t_i for each $i = 1, 2, \ldots, n$. Let $S_i \subseteq F$, $1 \le i \le n$, be a set of more than t_i distinct members in F. If $f(x_1, x_2, \ldots, x_n) = 0$ for all n-tuple $(x_1, x_2, \ldots, x_n) \in S_1 \times S_2 \times \cdots \times S_n$, then $f \equiv 0$.
Proof:

One proceeds by mathematical induction on n. The case where $n = 1$ is clear. Now suppose the lemma holds for $n-1$ variables. Given a polynomial $f = f(x_1, x_2, \ldots, x_n)$ and sets S_i's satisfying the hypothesis of the lemma, one writes

$$f = \sum_{r=0}^{t_n} f_r(x_1, x_2, \ldots, x_{n-1}) x_n^r \,.$$

For a fixed $(n-1)$-tuple $(x_1, x_2, \ldots, x_{n-1}) \in S_1 \times S_2 \times \cdots \times S_{n-1}$, the polynomial f is a polynomial in solely one variable x_n. Applying the case when $n = 1$, the fact that f vanishes for at least t_n+1 values (those elements in S_n) of x_n implies that the coefficients $f_r(x_1, x_2, \ldots, x_{n-1}) = 0$ for all r. This holds for any tuple $(x_1, x_2, \ldots, x_{n-1}) \in S_1 \times S_2 \times \cdots \times S_{n-1}$. Therefore, for each $r = 0, 1, \ldots, t_n$, $(x_1, x_2, \ldots, x_{n-1}) \in S_1 \times S_2 \times \cdots \times S_{n-1}$ implies $f_r(x_1, x_2, \ldots, x_{n-1}) = 0$. The induction hypothesis says that all f_r must be identically zero, and the lemma is justified.

$\square$

Lemma 2: Let F be an arbitrary field and f a polynomial in $F[x_1, x_2, \ldots, x_n]$. Given are nonempty finite subsets S_1, S_2, ..., S_n of F. Define $g_i(x_i) = \prod\limits_{s \in S_i} (x_i - s)$. If f vanishes over all the common zeros of $g_1, g_2, \ldots, g_n$ (that is, $f(s_1, s_2, \ldots, s_n) = 0$ for all $s_i \in S$), then there exist polynomials $h_1, h_2, \ldots, h_n \in F[x_1, x_2, \ldots, x_n]$ satisfying

- $\deg(h_i) \le \deg(f) - \deg(g_i)$, and
- $f = \sum\limits_{i=1}^{n} h_i g_i$.

Proof:

Define $t_i = |S_i| - 1$ for each $i = 1, 2, \ldots, n$. From the assumption, $f(x_1, x_2, \ldots, x_n) = 0$ for any n-tuple $(x_1, x_2, \ldots, x_n) \in S_1 \times S_2 \times \cdots \times S_n$. For each i, write

$$g_i(x_i) = x_i^{t_i+1} - \sum_{r=0}^{t_i} g_{i,r} x_i^r \,.$$

Thus, if $x_i \in S_i$, then

$$x_i^{t_i+1} = \sum_{r=0}^{t_i} g_{i,r} x_i^r \,.$$

Let $\tilde{f}$ be the polynomial obtained by writing f as a linear combination of monomials and replacing, repetedly, each occurence of $x_i^{f_i}$, $1 \le i \le n$, where $f_i > t_i$ by a linear combination of smaller power of x_i, using the relation $x_i^{t_i+1} = \sum\limits_{r=0}^{t_i} g_{i,r} x_i^r$.

Hence, $\tilde{f}$ is of degree at most t_i in x_i for every i and is obtained from f by subtracting from it the sum of the products of the form $h_i g_i$, where $h_i \in F[x_1, x_2, \ldots, x_n]$ has degree at most $\deg(f) - \deg(g_i)$. Since $f(x_1, x_2, \ldots, x_n) = \tilde{f}(x_1, x_2, \ldots, x_n)$ for all $(x_1, x_2, \ldots, x_n) \in S_1 \times S_2 \times \cdots \times S_n$, then,

$$\tilde{f}(x_1, x_2, \ldots, x_n) = 0 ,$$

for all $(x_1, x_2, \ldots, x_n) \in S_1 \times S_2 \times \cdots \times S_n$. Using Lemma 1, $\tilde{f} \equiv 0$ and hence,

$$f = \sum_{i=1}^{n} h_i g_i ,$$

as desired.

□

Lemma 3: (*Combinatorial Nullstellensatz*) Let F be an arbitrary field and f a polynomial in $F[x_1, x_2, \ldots, x_n]$. Suppose the degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer. Assume that the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ is nonzero. Then, if $S_1, S_2, \ldots, S_n$ are subsets of F with $|S_i| > t_i$, then there are $s_1 \in S_1$, $s_2 \in S_2$, ..., $s_n \in S_n$ so that

$$f(s_1, s_2, ..., s_n) \neq 0 .$$

Proof:

Clearly, one may set $|S_i| = t_i + 1$ for all i. Suppose the result is false and define $g_i(x_i) = \prod_{s \in S_i} (x_i - s)$. By Lemma 2, there are polynomials $h_1, h_2, \ldots, h_n \in F[x_1, x_2, \ldots, x_n]$ satisfying $\deg(h_i) \leq \deg(f) - \deg(g_i)$ and

$$f = \sum_{i=1}^{n} h_i g_i .$$

By assumption, the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ is not zero. Hence, the term $\prod_{i=1}^{n} x_i^{t_i}$ must occur somewhere on the right hand side. However, $h_i g_i = h_i \left(\prod_{s \in S_i} (x_i - s) \right)$ has degree of atmost $\deg(f)$. If there were any monomials of degree $\deg(f)$ in $h_i g_i$, they must be divisible by $x_i^{t_i+1}$. Thus, the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in the right hand side is zero, a contradiction!

□

Suppose that the smallest number of plane is ℓ. The following set of planes gives $\ell \leq 3n$:

$$\{(x, y, z) | z = 1, 2, \ldots, n\}$$

and

$$\{(x, y, z) | x + y = 1, 2, \ldots, 2n\} .$$

One must show that $\ell = 3n$ is the smallest possible number.

Assume contrary that $\ell < 3n$ and there is a set of planes $a_r x + b_r y + c_r z + d_r = 0$, $r = 1, 2, \ldots, \ell$, whose union contains S but does not include $(0, 0, 0)$. Consider the polynomial $P(x, y, z) \in \mathbb{R}[x, y, z]$ defined as follows:

$$P(x,y,z) = \prod_{r=1}^{\ell} (a_r x + b_r y + c_r z + d_r) - \delta \left(\prod_{i=1}^{n} (x-i) \right) \left(\prod_{j=1}^{n} (y-j) \right) \left(\prod_{k=1}^{n} (z-k) \right),$$

in which the constant δ is so chosen that $P(0,0,0) = 0$. (Clearly, $\delta \neq 0$.) Consequently, $P(x, y, z) = 0$ for all $x, y, z \in \{0, 1, 2, \cdots, n\}$. Since the degree of $\prod_{r=1}^{\ell} (a_r x + b_r y + c_r z + d_r)$ is ℓ which is less than $3n$, therefore, the term $x^n y^n z^n$ does not occur in the expansion of this product. In $\delta \left(\prod_{i=1}^{n} (x-i) \right) \left(\prod_{j=1}^{n} (y-j) \right) \left(\prod_{k=1}^{n} (z-k) \right)$, it is clear that the coefficient of $x^n y^n z^n$ is δ which does not equal zero.

According to Combinatorial Nullstellensatz (Lemma 3), let $S_x = S_y = S_z = \{0, 1, 2, \ldots, n\}$ (hence, $|S_x| = |S_y| = |S_z| = n + 1$). Therefore, $P(x, y, z)$ vanishes for all $x \in S_x$, $y \in S_y$, $z \in S_z$. This means the coefficient of $x^n y^n z^n$ must be zero, which has been proved that it is not. Therefore, $\ell = 3n$, as desired.